October Math Gems

Problem of the week 33

§1 Problems

Problem 1.1. a, b, c, d satisfy the following system of equations

$$ab + c + d = 13$$

$$bc + d + a = 27$$

$$cd + a + b = 30$$

$$da + b + c = 17$$

Compute the value of $a + b + c + d$.

Solution. □

Problem 1.2. Suppose that we have the following set of equations

$$\log_2 x + \log_3 x + \log_4 x = 20$$

$$\log_4 y + \log_9 y + \log_{16} y = 16$$

Compute $\log_x y$.

Solution. □

Problem 1.3. If the function f satisfy the following relation

$$f\left(x + \frac{1}{x}\right) = x^3 + \frac{1}{x^3}$$

Then compute $f(4)$

Solution. We start seeking the solution by noticing the following:

$$x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right)$$

We can substitute with this in the original function yielding:

$$f\left(x + \frac{1}{x}\right) = \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right)$$

As we need to compute $f(4)$, let $x + \frac{1}{x} = 4$.

$$f(4) = (4)^3 - 3(4)$$

$$f(4) = 52$$

□

Problem 1.4. In the following functional equation, solve for $f(x)$

$$f\left(x+\sqrt{x^2+1}\right)=\frac{x}{x+1}$$

Solution. In this type of problems the easiest approach is to use substitution, so let:

$$x+\sqrt{x^2+1}=t \quad \textit{[1]}$$

This yields:

$$f(t)=\frac{x}{x+1}$$

Now, we need the function to be in terms of t, so we need to solve for x in *[1]*, by doing so we get:

$$x=\frac{t^2-1}{2t}$$

Then we'll substitute with this in the function:

$$f(t)=\frac{\frac{t^2-1}{2t}}{\frac{t^2-1}{2t}+1}$$

All the left is to do some simple algebraic manipulations.

$$f(t)=\frac{t^2-1}{t^2+2t-1}$$

Notice that there is no difference between x and t, they are just variables.

$$f(x)=\frac{x^2-1}{x^2+2x-1}$$

□

Problem 1.5. Let a, b, and c be distinct nonzero real numbers such that

$$a+\frac{1}{b}=b+\frac{1}{c}=c+\frac{1}{a}$$

Prove that $|abc|=1$

Solution. From the given, we can find that:

$$a-b=\frac{b-c}{bc}$$

$$b-c=\frac{c-a}{ca}$$

$$c-a=\frac{a-b}{ab}$$

Multiplying the above equations gives us $(abc)^2=1$, which proves that $|abc|=1$. □

Problem 1.6. Find all real numbers x for which

$$\frac{8^x+27^x}{12^x+18^x}=\frac{7}{6}$$

Solution. Let $2^x = a$ and $3^x = b$, then we can rewrite the equation as following:

$$\frac{a^3+b^3}{a^2b+ab^2} = \frac{7}{6}$$

Notice that we can factor out ab from the denominator of the left-hand side.

$$\frac{a^3+b^3}{ab(a+b)} = \frac{7}{6}$$

In order to get the most simpler form, we'll use the following fact: $(a^3+b^3) = (a+b)(a^2-ab+b^2)$

$$\begin{aligned}
&= \frac{(a+b)(a^2-ab+b^2)}{ab(a+b)} = \frac{7}{6} \\
&= \frac{a^2-ab+b^2}{ab} = \frac{7}{6} \\
&= 6a^2 - 13ab + 6b^2 = 0 \\
&= (2a-3b)(3a-2b) = 0
\end{aligned}$$

Therefore we have two solutions $2^{x+1} = 3^{x+1}$ or $2^{x-1} = 3^{x-1}$, which implies that x can be equal to 1 or -1. □

Problem 1.7. Find all real numbers x satisfying the equation

$$2^x + 3^x - 4^x + 6^x - 9^x = 1$$

Solution. Let $2^x = a$ and $3^x = b$, so we can rewrite the equation as following:

$$1 + a^2 + b^2 - a - b - ab = 0$$

Now we'll multiply both sides by 2 and complete the squares yielding the following:

$$(1-a)^2 + (a-b)^2 + (b-1)^2 = 0$$

This means that $1 = 2^x = 3^x$, so $x = 0$ is the only solution. □

Problem 1.8. If $z = x - iy$ and $z^{\frac{1}{3}} = p + iq$, then compute

$$\frac{x/p + y/q}{p^2+q^2}$$

.

Solution. To get to a more manageable form of the following equation

$$z^{\frac{1}{3}} = p + iq$$

We'll raise both sides to the power of 3, yielding:

$$z = (p+iq)^3$$

Recall that z is equivalent $x - iy$, so:

$$x - iy = (p+iq)^3$$

Now, we'll expand the right-hand side.

$$x - iy = p^3 - 3pq^2 + \left(3p^2q - q^3\right) i$$

Then, we'll equate real and imaginary parts.

$$x = p^3 - 3pq^2$$

$$\frac{x}{p} = p^2 - 3q^2 \qquad \textit{[1]}$$

For the imaginary part:

$$-y = 3p^2q - q^3$$

$$\frac{y}{q} = q^2 - 3p^2 \qquad \textit{[2]}$$

By adding *[1]* and *[2]*, we get:

$$\frac{x}{p} + \frac{y}{q} = -2(p^2 + q^2)$$

$$\frac{x/p + y/q}{p^2 + q^2} = -2$$

□

Problem 1.9. If

$$\frac{\log(a)}{b-c} = \frac{\log(b)}{c-a} = \frac{\log(c)}{a-b}$$

Then compute $a^a b^b c^c$.

Solution. Let:

$$\frac{\log(a)}{b-c} = \frac{\log(b)}{c-a} = \frac{\log(c)}{a-b} = k$$

So we can get the following:

$$\log(a) = k(b-c)$$

$$\log(b) = k(c-a)$$

$$\log(c) = k(a-b)$$

Now:

$$\begin{aligned} \log(a^a b^b c^c) &= a\log(a) + b\log(b) + c\log(c) \\ &= a\left(k\left(b-c\right)\right) + b\left(k\left(c-a\right)\right) + c\left(k\left(a-b\right)\right) \\ &= 0 \end{aligned}$$

This means that $a^a b^b c^c = 1$

□

Problem 1.10. If the coefficients of x^{-2} and x^{-4} in the expansion of

$$\left(x^{\frac{1}{3}} + \frac{1}{2x^{\frac{1}{3}}}\right) \qquad (x > 0)$$

are m and n respectively, then compute $\frac{m}{n}$.

Solution. First, we need to find the general term:

$$T_{r+1} = [18]r \left(x^{\frac{1}{3}}\right)^{18-r} \left(\frac{1}{2x^{\frac{1}{3}}}\right)^{r}$$
$$= [18]r x^{6-\frac{2r}{3}} \frac{1}{2^r}$$

So, to get the coefficients of x^{-2} we need to assume:

$$6 - \frac{2r}{3} = -2$$
$$r = 12$$

Doing the same for x^{-4}

$$6 - \frac{2r}{3} = -4$$
$$r = 15$$

$\frac{\text{coefficient of } x^{-2}}{\text{coefficient of } x^{-4}} = \frac{[18]12\frac{1}{2^{12}}}{[18]15\frac{1}{2^{15}}} = 182$

□

Problem 1.11. The coefficients of x^{50} in the expansion of

$$(1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \cdots + 1001x^{1000}$$

is [1002]50. Prove or Disprove.

Solution. Let:

$$S = (1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \cdots + 1001x^{1000} \quad \textit{[1]}$$

$$\frac{x}{1+x}S = x(1+x)^{999} + 2x^2(1+x)^{998} + \cdots + 1000x^{1000} + \frac{1001x^{1001}}{1+x} \quad \textit{[2]}$$

Now, subtract *[1]* and *[2]* to get the following:

$$\left(1 - \frac{x}{1+x}\right) S = (1+x)^{1001} + x(1+x)^{999} + \cdots + x^{1000} - \frac{1001x^{1001}}{1+x}$$

$$S = (1+x)^{1001} + x(1+x)^{1000} + x^2(1+x)^{999} + \cdots + x^{1000} - 1001x1001$$

Notice that this is a sum of geometric pattern.

$$S = (1+x)^{1002} - x^{1002} - 1002x^{1001})$$

So the coefficient of x^{50} is [1002]50 □

Problem 1.12. Suppose x and y are nonzero real numbers simultaneously satisfying the following system of equations

$$x + \frac{2018}{y} = 1000$$

$$\frac{9}{x} + y = 1$$

Find the maximum possible value of $x + 1000y$.

Solution. First, we need to multiply the first equation with y, and the second with x to obtain the following:

$$xy + 2018 = 1000y$$

$$9 + xy = x$$

Subtracting the two equations yielding:

$$2009 = 1000y - x$$

Now, we need to solve the above equation for y, then substitute it into $9 + xy = x$ yields:

$$x^2 + 1009x + 9000 = 0$$

which factors as:

$$(x + 9)(x + 1000) = 0$$

This gives us two possible solutions

$$(x, y) = (9, 2)$$

$$(x, y) = (-1000, \frac{1009}{1000})$$

Then the requested sum is $-9 + 1000 \cdot 2 = 1991$ □

Problem 1.13. From the following system of equations

$$x^2 - y^2 = 9$$

$$xy = 3$$

The value of of $x + y$ can be written in the form of $\pm\sqrt{\sqrt{a} + b}$, then find the values of a and b.

Solution. First, we need to solve for y in the second equation and substitute in the first one yielding:

$$x^2 - \left(\frac{3}{x}\right)^2 = 9$$

$$x^4 - 9x^2 - 9 = 0$$

$$x^2 = \frac{9 + \sqrt{117}}{2}$$

$$x = \pm\sqrt{\frac{9 + \sqrt{117}}{2}}$$

Thus, we can use this in finding the value of y.

$$y = \pm\sqrt{\frac{9 - \sqrt{117}}{2}}$$

So we get that,

$$x + y = \pm\left(\sqrt{\frac{9 + \sqrt{117}}{2}} + \sqrt{\frac{9 - \sqrt{117}}{2}}\right)$$

Unfortunately, we're not done yet, we still need to acquire the form of $\sqrt{\sqrt{a}+b}$ which requires us to square both sides, yielding:

$$(x+y)^2 = \sqrt{117}+6$$

$$x+y = \pm\sqrt{\sqrt{117}+6}$$

Thus, $a = 117$, $b = 6$. $\square$

Problem 1.14. Determine the domain of the function

$$g(x) = \cot^{-1}\left(\frac{x}{\sqrt{x^2 - \lfloor x^2 \rfloor}}\right)$$

Solution. For $g(x)$ to be defined,

$$x^2 - \lfloor x^2 \rfloor > 0$$

Thus, x^2 cannot be integer, also 0 is restricted.

We need to use the fact that:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Hence,

$$\begin{aligned}\left|\frac{a+bi}{b+ai}\right| &= \frac{|a+bi|}{|b+ai|}\\ &= \frac{\sqrt{a^2+b^2}}{b^2+a^2}\\ &= 1\end{aligned}$$

$\square$

Problem 1.15. If a and b are two real numbers, then show that

$$\left|\frac{a+bi}{b+ai}\right| = 1$$

Solution. For $f(x)$ to be defined,

$$x^2 - \lfloor x^2 \rfloor > 0$$

Thus, x^2 cannot be integer, also 0 is restricted.

We need to use the fact that:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Hence,

$$\begin{aligned}\left|\frac{a+bi}{b+ai}\right| &= \frac{|a+bi|}{|b+ai|}\\ &= \frac{\sqrt{a^2+b^2}}{b^2+a^2}\\ &= 1\end{aligned}$$

$\square$

Problem 1.16. Let a, b, c be distinct real numbers. Prove the the following equality cannot hold:

$$\sqrt[3]{a-b} + \sqrt[b]{b-c} + \sqrt[3]{c-a} = 0$$

Solution. First, we'll assume that the inverse of our claim is correct, thus:

$$\sqrt[3]{a-b} + \sqrt[3]{b-c} + \sqrt[3]{c-a} = 0$$

By raising the both sides to the power of 3, we'll get the following:

$$(a-b) + (b-c) + (c-a) = 3\sqrt[3]{(a-b) + (b-c) + (c-a)}$$

This implies that:

$$(a-b) + (b-c) + (c-a) = 0$$

which contradicts that a, b, c are distinct. $\square$

Problem 1.17. let r be a real number such that

$$\sqrt[3]{r} + \frac{1}{\sqrt[3]{r}} = 3$$

Determine the value of

$$r^3 + \frac{1}{r^3}$$

Note that by raising both sides to the power of 3, that we get:

$$r + \frac{1}{r} - 18 = 0$$

By doing the same in the last step, we get:

$$r^3 + \frac{1}{r^3} = 5778$$

Problem 1.18. let $x, y, z > 0$. Prove that

$$\frac{2}{x+y} + \frac{2}{y+z} + \frac{2}{z+x} \geq \frac{9}{x+y+z}$$

Solution. We'll rewrite the left-hand side a little bit as following:

$$\frac{(\sqrt{2})^2}{x+y} + \frac{(\sqrt{2})^2}{y+z} + \frac{(\sqrt{2})^2}{z+x}$$

and we'll use the following lemma:

Lemma 1.19

If a, b, x, y are real numbers and $x, y > 0$, then the following inequality holds:

$$\frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}$$

We deduce that:

$$\frac{(\sqrt{2})^2}{x+y} + \frac{(\sqrt{2})^2}{y+z} + \frac{(\sqrt{2})^2}{z+x} \geq \frac{(3\sqrt{2})^2}{2(x+y+z)} = \frac{9}{x+y+z}$$

□

Problem 1.20. Let m, n be positive integers with $m < n$. Find the a closed form for the sum

$$\frac{1}{\sqrt{m}+\sqrt{m+1}} + \frac{1}{\sqrt{m+1}+\sqrt{m+2}} + \cdots + \frac{1}{\sqrt{n-1}+\sqrt{n}}$$

Solution. By taking the conjugate for each term of the sum, we get the following:

$$\frac{\sqrt{m+1}-\sqrt{m}}{m+1-m} + \frac{\sqrt{m+2}-\sqrt{m+1}}{m+2-m-1} + \cdots + \frac{\sqrt{n}-\sqrt{n-1}}{n-n+1}$$

which is equal to:

$$\sqrt{m+1}-\sqrt{m}+\sqrt{m+2}-\sqrt{m+1}+\cdots+\sqrt{n}-\sqrt{n-1} = \sqrt{n}-\sqrt{m}$$

. □

Problem 1.21. Let a and b be distinct real numbers. Solve the following equation

$$\sqrt{x-b^2} - \sqrt{x-a^2} = a-b$$

Solution. It should be obvious that the following conditions must hold true:

$$x \geq a^2 \quad x \geq b^2$$

Actually the simplest approach to solve this equation is taking the conjugate, another approaches leads to rather complicated computations. Taking the conjugate gives

$$\frac{a^2-b^2}{\sqrt{x-b^2}+\sqrt{x-a^2}} = a-b$$

which is equivalent to

$$\sqrt{x-b^2} + \sqrt{x-a^2} = a+b$$

Adding this to the original equation gives the following:

$$\sqrt{x-b^2} = a$$

This implies that

$$x = \sqrt{a^2+b^2}$$

□
